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Research Article

Strong Convergence of Monotone Hybrid Algorithm for Hemi-Relatively Nonexpansive Mappings

Yongfu Su,[1] Dongxing Wang,[1] and Meijuan Shang[1,2]

[1] *Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China*
[2] *Department of Mathematics, Shijiazhuang University, Shijiazhuang 050035, China*

Correspondence should be addressed to Yongfu Su, suyongfu@tjpu.edu.cn

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The purpose of this article is to prove strong convergence theorems for fixed points of closed hemi-relatively nonexpansive mappings. In order to get these convergence theorems, the monotone hybrid iteration method is presented and is used to approximate those fixed points. Note that the hybrid iteration method presented by S. Matsushita and W. Takahashi can be used for relatively nonexpansive mapping, but it cannot be used for hemi-relatively nonexpansive mapping. The results of this paper modify and improve the results of S. Matsushita and W. Takahashi (2005), and some others.

1. Introduction

In 2005, Shin-ya Matsushita and Wataru Takahashi [1] proposed the following hybrid iteration method (it is also called the CQ method) with generalized projection for relatively nonexpansive mapping T in a Banach space E:

$$\begin{aligned}
x_0 &\in C \quad \text{chosen arbitrarily},\\
y_n &= J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)JTx_n\big),\\
C_n &= \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\},\\
Q_n &= \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\},\\
x_{n+1} &= \prod_{C_n \cap Q_n}(x_0).
\end{aligned} \tag{1.1}$$

They proved the following convergence theorem.

Theorem 1.1 (MT). *Let E be a uniformly convex and uniformly smooth real Banach space, let C be a nonempty, closed, and convex subset of E, let T be a relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\limsup_{n\to\infty}\alpha_n < 1$. Suppose that $\{x_n\}$ is given by* (1.1)*, where J is the duality mapping on E. If the set $F(T)$ of fixed points of T is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}(\cdot)$ is the generalized projection from C onto $F(T)$.*

The purpose of this article is to prove strong convergence theorems for fixed points of closed hemi-relatively nonexpansive mappings. In order to get these convergence theorems, the monotone hybrid iteration method is presented and is used to approximate those fixed points. Note that the hybrid iteration method presented by S.Matsushita and W. Takahashi can be used for relatively nonexpansive mapping, but it cannot be used for hemi-relatively nonexpansive mapping. The results of this paper modify and improve the results of S.Matsushita and W. Takahashi [1], and some others.

2. Preliminaries

Let E be a real Banach space with dual E^* . We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \tag{2.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. It is well known that if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E. In this case, J is singe valued and also one to one.

Recall that if C is a nonempty, closed, and convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This is true only when H is a real Hilbert space. In this connection, Alber [2] has recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that E is a smooth Banach space. Consider the functional defined as [2, 3] by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for } x, y \in E. \tag{2.2}$$

Observe that, in a Hilbert space H, (2.2) reduces to $\phi(x, y) = \|x - y\|^2$, $x, y \in H$.

The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \min_{y\in C}\phi(y, x), \tag{2.3}$$

existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping J (see, e.g., [2–4]). In Hilbert space, $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2 \quad \forall x, y \in E. \tag{2.4}$$

Remark 2.1. If E is a reflexive strict convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$, then $x = y$. From (2.4), we have $\|x\| = \|y\|$. This implies $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, we have $Jx = Jy$, that is, $x = y$; see [5] for more details.

We refer the interested reader to the [6], where additional information on the duality mapping may be found.

Let C be a closed convex subset of E, and Let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. T is called hemi-relatively nonexpansive if $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

A point p in C is said to be an asymptotic fixed point of T [7] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(Tx_n - x_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A hemi-relatively nonexpansive mapping T from C into itself is called relatively nonexpansive [1, 7, 8] if $\widehat{F}(T) = F(T)$.

We need the following lemmas for the proof of our main results.

Lemma 2.2 (Kamimura and Takahashi [4], [1, Proposition 2.1]). *Let E be a uniformly convex and smooth real Banach space and let $\{x_n\}$, $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 2.3 (Alber [2], [1, Proposition 2.2]). *Let C be a nonempty closed convex subset of a smooth real Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\left\langle x_0 - y, Jx - Jx_0\right\rangle \ge 0 \quad \forall y \in C. \tag{2.5}$$

Lemma 2.4 (Alber [2], [1, Proposition 2.3]). *Let E be a reflexive, strict convex, and smooth real Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi\left(y, \prod_C x\right) + \phi\left(\prod_C x, x\right) \le \phi(y, x) \quad \forall y \in C. \tag{2.6}$$

By using the similar method as [1, Proposition 2.4], the following lemma is not hard to prove.

Lemma 2.5. *Let E be a strictly convex and smooth real Banach space, let C be a closed convex subset of E, and let T be a hemi-relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

Recall that an operator T in a Banach space is called closed, if $x_n \to x$, $Tx_n \to y$, then $Tx = y$.

3. Strong convergence for hemi-relatively nonexpansive mappings

Theorem 3.1. *Theorem 3.1 Let E be a uniformly convex and uniformly smooth real Banach space, let C be a nonempty closed convex subset of E, let $T : C \to C$ be a closed hemi-relatively nonexpansive mapping such that $F(T) \ne \emptyset$. Assume that $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\limsup_{n\to\infty}\alpha_n < 1$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{aligned}
&x_0 \in C \quad \textit{chosen arbitrarily},\\
&y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n),\\
&C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \le \phi(z, x_n)\},\\
&C_0 = \{z \in C : \phi(z, y_0) \le \phi(z, x_0)\},\\
&Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, J x_0 - J x_n\rangle \ge 0\},\\
&Q_0 = C,\\
&x_{n+1} = \prod_{C_n \cap Q_n}(x_0),
\end{aligned} \tag{3.1}$$

where J is the duality mapping on E. Then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

Proof. We first show that C_n and Q_n are closed and convex for each $n \ge 0$. From the definition of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for each $n \ge 0$. We show that C_n is convex for any $n \ge 0$. Since

$$\phi(z, y_n) \le \phi(z, x_n) \tag{3.2}$$

is equivalent to

$$2\langle z, J x_n - J y_n\rangle \le \|x_n\|^2 - \|y_n\|^2, \tag{3.3}$$

it follows that C_n is convex.

Next, we show that $F(T) \subset C_n$ for all $n \ge 0$. Indeed, we have for all $p \in F(T)$ that

$$\begin{aligned}
\phi(p, y_n) &= \phi(p, j^{-1}(\alpha_n j x_n + (1-\alpha_n) j t x_n))\\
&\le \|p\|^2 - 2\langle p, \alpha_n j x_n + (1-\alpha_n) j t x_n\rangle + \alpha_n \|x_n\|^2 + (1-\alpha_n)\|t x_n\|^2\\
&= \alpha_n \phi(p, x_n) + (1-\alpha_n)\phi(p, t x_n)\\
&\le \alpha_n \phi(p, x_n) + (1-\alpha_n)\phi(p, x_n)\\
&= \phi(p, x_n).
\end{aligned} \tag{3.4}$$

That is, $p \in C_n$ for all $n \ge 0$.

Next, we show that $F(T) \subset Q_n$ for all $n \ge 0$, we prove this by induction. For $n = 0$, we have $F(T) \subset C = Q_0$. Assume that $F(T) \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, by Lemma 2.3, we have

$$\langle x_{n+1} - z, J x_0 - J x_{n+1}\rangle \ge 0, \quad \forall z \in C_n \cap Q_n. \tag{3.5}$$

As $F(T) \subset C_n \cap Q_n$ by the induction assumptions, the last inequality holds, in particular, for all $z \in F(T)$. This together with the definition of Q_{n+1} implies that $F(T) \subset Q_{n+1}$.

Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0$ and $C_n \cap Q_n \subset C_{n-1} \cap Q_{n-1}$ for all $n \ge 1$, we have

$$\phi(x_n, x_0) \le \phi(x_{n+1}, x_0) \tag{3.6}$$

for all $n \ge 0$. Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing. In addition, it follows from the definition of Q_n and Lemma 2.3 that $x_n = \Pi_{Q_n} x_0$. Therefore, by Lemma 2.4, we have

$$\phi(x_n, x_0) = \phi\Big(\prod_{Q_n} x_0, x_0\Big) \le \phi(p, x_0) - \phi(p, x_n) \le \phi(p, x_0), \tag{3.7}$$

for each $p \in F(T) \subset Q_n$ for all $n \geq 0$. Therefore, $\phi(x_n, x_0)$ is bounded, this together with (3.6) implies that the limit of $\{\phi(x_n, x_0)\}$ exists. Put

$$\lim_{n\to\infty} \phi(x_n, x_0) = d. \tag{3.8}$$

From Lemma 2.4, we have, for any positive integer m, that

$$\begin{aligned}\phi(x_{n+m}, x_n) &= \phi\Big(x_{n+m}, \prod_{C_n} x_0\Big)\\ &\leq \phi(x_{n+m}, x_0) - \phi\Big(\prod_{C_n} x_0, x_0\Big) = \phi(x_{n+m}, x_0) - \phi(x_n, x_0),\end{aligned} \tag{3.9}$$

for all $n \geq 0$. Therefore,

$$\lim_{n\to\infty} \phi(x_{n+m}, x_n) = 0. \tag{3.10}$$

We claim that $\{x_n\}$ is a Cauchy sequence. If not, there exists a positive real number $\varepsilon_0 > 0$ and subsequence $\{n_k\}, \{m_k\} \subset \{n\}$ such that

$$\left\| x_{n_k+m_k} - x_{n_k} \right\| \geq \varepsilon_0, \tag{3.11}$$

for all $k \geq 1$.

On the other hand, from (3.8) and (3.9) we have

$$\begin{aligned}\phi(x_{n_k+m_k}, x_{n_k}) &\leq \phi(x_{n_k+m_k}, x_0) - \phi(x_{n_k}, x_0)\\ &\leq \left|\phi(x_{n_k+m_k}, x_0) - d\right| + |d - \phi(x_{n_k}, x_0)| \longrightarrow 0, \quad k \longrightarrow \infty.\end{aligned} \tag{3.12}$$

Because from (3.8) we know that $\phi(x_n, x_0)$ is bounded, this and (2.4) imply that $\{x_n\}$ is also bounded, so by Lemma 2.2 we obtain

$$\lim_{k\to\infty} \left\| x_{n_k+m_k} - x_{n_k} \right\| = 0. \tag{3.13}$$

This is a contradiction, so that $\{x_n\}$ is a Cauchy sequence, therefore there exists a point $p \in C$ such that $\{x_n\}$ converges strongly to p.

Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in C_n$, from the definition of C_n, we have

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n). \tag{3.14}$$

It follows from (3.10), (3.14) that

$$\phi(x_{n+1}, y_n) \longrightarrow 0. \tag{3.15}$$

By using Lemma 2.2, we have

$$\lim_{n\to\infty} \left\| x_{n+1} - y_n \right\| = \lim_{n\to\infty} \left\| x_{n+1} - x_n \right\| = 0. \tag{3.16}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}\|Jx_{n+1}-Jy_n\| = \lim_{n\to\infty}\|Jx_{n+1}-Jx_n\| = 0. \tag{3.17}$$

Noticing that

$$\begin{aligned}\|Jx_{n+1}-Jy_n\| &= \|Jx_{n+1}-(\alpha_n Jx_n+(1-\alpha_n)JTx_n)\| \\ &= \|\alpha_n(Jx_{n+1}-Jx_n)+(1-\alpha_n)(Jx_{n+1}-JTx_n)\| \\ &= \|(1-\alpha_n)(Jx_{n+1}-Jtx_n)-\alpha_n(Jx_n-Jx_{n+1})\| \\ &\geq (1-\alpha_n)\|Jx_{n+1}-Jtx_n\|-\alpha_n\|Jx_n-Jx_{n+1}\|,\end{aligned} \tag{3.18}$$

which implies that

$$\|Jx_{n+1}-JTx_n\| \leq \frac{1}{1-\alpha_n}\left(\|Jx_{n+1}-Jy_n\|+\alpha_n\|Jx_n-Jx_{n+1}\|\right). \tag{3.19}$$

This together with (3.17) and $\limsup_{n\to\infty}\alpha_n<1$ implies that

$$\lim_{n\to\infty}\|Jx_{n+1}-JTx_n\| = 0. \tag{3.20}$$

Since J^{-1} is also uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty}\|x_{n+1}-Tx_n\| = 0. \tag{3.21}$$

Observe that

$$\|x_n-Tx_n\| \leq \|x_n-x_{n+1}\|+\|x_{n+1}-Tx_n\|. \tag{3.22}$$

It follows from (3.16) and (3.21) that

$$\lim_{n\to\infty}\|x_n-Tx_n\| = 0. \tag{3.23}$$

Since T is a closed operator and $x_n\to p$, then p is a fixed point of T.

Finally, we prove that $p=\Pi_{F(T)}x_0$. From Lemma 2.4, we have

$$\phi\Big(p,\prod_{F(T)}x_0\Big)+\phi\Big(\prod_{F(T)}x_0,x_0\Big) \leq \phi(p,x_0). \tag{3.24}$$

On the other hand, since $x_{n+1}=\Pi_{C_n\cap Q_n}$ and $C_n\cap Q_n\supset F(T)$, for all n, we get from Lemma 2.4 that

$$\phi\Big(\prod_{F(T)}x_0,x_{n+1}\Big)+\phi(x_{n+1},x_0) \leq \phi\Big(\prod_{F(T)}x_0,x_0\Big). \tag{3.25}$$

By the definition of $\phi(x,y)$, it follows that both $\phi(p,x_0)\leq\phi(\Pi_{F(T)}x_0,x_0)$ and $\phi(p,x_0)\geq\phi(\Pi_{F(T)}x_0,x_0)$, whence $\phi(p,x_0)=\phi(\Pi_{F(T)}x_0,x_0)$. Therefore, it follows from the uniqueness of $\Pi_{F(T)}x_0$ that $p=\Pi_{F(T)}x_0$. This completes the proof. □

Theorem 3.2. *Let E be a uniformly convex and uniformly smooth real Banach space, let C be a nonempty, closed, and convex subset of E, and let $T : C \to C$ be a closed relative nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ is a sequences in $[0,1]$ such that* $\limsup_{n\to\infty}\alpha_n < 1$. *Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{aligned}
x_0 &\in C \quad \text{chosen arbitrarily},\\
y_n &= J^{-1}\left(\alpha_n J x_n + (1-\alpha_n) J T x_n\right),\\
C_n &= \left\{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\right\},\\
C_0 &= \left\{z \in C : \phi(z, y_0) \leq \phi(z, x_0)\right\},\\
Q_n &= \left\{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, J x_0 - J x_n \rangle \geq 0\right\},\\
Q_0 &= C,\\
x_{n+1} &= \prod_{C_n \cap Q_n} (x_0),
\end{aligned} \tag{3.26}$$

where J is the duality mapping on E. Then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

Proof. Since every relatively nonexpansive mapping is a hemi-relatively one, Theorem 3.2 is implied by Theorem 3.1. □

Remark 3.3. In recent years, the hybrid iteration methods for approximating fixed points of nonlinear mappings have been introduced and studied by various authors [1, 8–11]. In fact, all hybrid iteration methods can be replaced (or modified) by monotone hybrid iteration methods, respectively. In addition, by using the monotone hybrid method we can easily show that the iteration sequence $\{x_n\}$ is a Cauchy sequence, without the use of the Kadec-Klee property, demiclosedness principle, and Opial's condition or other methods which make use of the weak topology.

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